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Primary Article

Properties Of Geometric Functions Theory With Negative Coefficient Of Multivalent Functions

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ABSTRACT:

In the geometric functions theory there are many subclasses of multivalent functions like coefficient bounds, radii of star likeness, convexity, harmonic, memomorphic functions, etc. Now we introduce some properties of geometric functions theory with negative coefficient of multivalent functions

$f(z) = z^p - \sum_{z=n}^{\infty} a_k z^{k+p}$**Which is analytic in unit disc**$\cup = \{z : |z| < 1\}$

Keyword:-

Analytic, memomorphic, multivalent functions, negative coefficient, geometric functions, distortion,hypergeometric function,pochhammer symbol, gamma function, Generalized Ruscheweyh derivatives.

Introduction:

Let, $f(z)$ **is the analytic and multivalent function belonging with**
$(n, p \in IN, a_k \geq 0)$ **Where,**

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots 1$$

$(a_k \geq 0; n \in IN)$

Where $\cup = \{z \in \mathbb{C}; |z| < 1\}$ **..........................2**

$f(z)$**is the function which belongs to** $s(n, p, \propto)$ **is** $\propto$ **ordered multivalent star like function.**

$f(z) \in T(n, p)$**Denote class of function,**iff,

$$R_e = \left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \propto \qquad \ldots\ldots\ldots\ldots\ldots\ldots 3$$

$(z \in \cup, 0 \leq \propto < p)$

$f(z) \in \mathbb{C}(n, p, \propto)$**When**

$zf'(z) \in s(n, p, \propto)$**For all** $n \in IN$ $\ldots\ldots\ldots\ldots\ldots\ldots\ldots 4$

By using Guass's hyper geometric function $a, b, c \in \mathbb{C}$ **with** $c \neq 0, -1, -2, \ldots\ldots\ldots\ldots$

$${}_2F_1(z) = {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a_n)(b)_n}{(c)_n} \frac{z^n}{n!} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 5$$

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Where λ_n is pochhammer symbol given by,

$$\lambda_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\dots\dots\dots(\lambda+n-1) & n \in IN \end{cases} \quad \dots\dots\dots\dots\dots 6$$

This is the form of Gamma function

Definition1:- If function $f(z) \in A(n)$ then class

$$A_\gamma^{\lambda,\mu,\nu}(n,\beta) = \{F \in A(n): R_e\left\{\frac{z(\mathcal{G}_1^{\lambda,\mu}f(z))'}{(1-\gamma)\mathcal{G}_1^{\lambda,\mu}f(z) + \gamma z^2(\mathcal{G}_1^{\lambda,\mu}f(z))^n}\right\} > \beta, (z \in \cup, 0 \le \nu < 1, n \in IN; 0 \le \beta < 1; \lambda - 1)\} \quad \dots\dots 1.1$$

As, $\mathcal{G}_1^{\lambda,\mu}f(z) = \frac{\Gamma(\mu-\lambda+\nu+2)}{\Gamma(\nu+2)\Gamma(\mu+1)} zJ_{0,z}^{\lambda,\mu,\nu}(z^{\mu-1}f(z))$; **where** $\mathcal{G}_1^{\lambda,\mu}f(z)$ **is generalized Ruschewegh derivatives**

$$= z - \sum_{k=n+1}^{\infty} a_k c_1^{\lambda,\mu}(k) z^k \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots 1.2$$

And,

$$c_1^{\lambda,\mu}(k) = \frac{\Gamma(k+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu+1)}{\Gamma(k)\Gamma(k+\nu+1+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots 1.3$$

Where $\mu = \lambda = \alpha, \nu = 1$ **Ruschewegh derivatives reduce in ordinary derivatives of** $f(z)$ **of order** $\propto$

$$D^n f(z) = \frac{z}{\Gamma(\propto+1)} D^{\propto}\left(z^{\propto-1}f(z)\right) = z - \sum_{k=n+1}^{\infty} a_k c_k(\propto) z_k \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots 1.4$$

Also,

$$c_k(\propto) = \frac{(\propto+1)(\propto+2)\dots\dots(\propto+k-1)}{(k-1)!} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots 1.5$$

The above equation analytic functions derived by class $A_\gamma^{\lambda,\mu,\nu}(n,\beta)$

- **(a) If** $\mu = \lambda = \alpha, \nu = 1, n = 1$ **the** $A_\gamma^{\lambda,\lambda,\nu}(1,\beta)$ **studies of the class of UF with negative coefficient defines by Ruschewegh [4]**
- **(b) If** $\mu = \lambda = 0, \nu = 1, \propto = \beta, \gamma = 0$ **getting star like functions of order** $\propto$ $(s(n,\propto))$.

Definition: 2 Let, F, h **be analytical in U. When h is subordinate to f, if** $\ni$ w **function that is** $|w(z)| < |$ **in U and** $h(z) = f(w(z))$ **in U for another analytic function of with w(0)=0.**

Definition: 3 Let, $0 \le \lambda \le 1$**&**$\mu, \nu \in IR$ **then generalized fractional derivatives operator** λ **is defined in terms of** $_2F_1$ **is,**

$$J_{0,z}^{\lambda,\mu,\nu}f(z) = \{\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\{z^{\lambda-\mu}\int_0^z (z-\epsilon)^{-\lambda} f(\varepsilon).\,{}_2F_1(\mu-\lambda, 1-\nu; 1-\lambda; 1-{}^{\varepsilon}/_{2})d\varepsilon\} \qquad 0 \le \lambda < 1$$

$$\frac{d^n}{dz^n} J_{0,z}^{\lambda-n,\mu,\nu} f(z); (n \le \lambda < n+1, n \in IN) \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots 3.1$$

Where f(z) is analytic in z-plane with origin simply- connected region of order

$$f(z) = 0(|z|)^{\varepsilon}(z \to 0) \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots 3.2$$

For $\varepsilon > \max\{0, \mu-\nu\} - 1$**&multiplicity of** $(z-\varepsilon)^{-\lambda}$ **is removed by requiring** $\log(z-\varepsilon)$ **to be real, when** $z - \varepsilon > 0$

Then, f(z) defined by,

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$D_z^{\lambda}\{f(z)\} = \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(\varepsilon)}{(z-\varepsilon)^{\lambda}}d\varepsilon; 0 \le \lambda < 1$ **Is a fractional derivative of order** λ**...........................3.3**

Where, f(z) from (3.1) and term $(z-\varepsilon)^{-\lambda}$ **is neglected by taking** $\log(z-\epsilon)$ **but** $z-\varepsilon > 0$ **with comparing (3.1) with (3.3) we find,**

$$J_{0,z}^{\lambda,\mu,v} f(z) = D_z^{\lambda}\{f(z)\} \; ; (0 \le \lambda < 1) \qquad \ldots 3.4$$

In the form of gamma function,

$$J_{0,z}^{\lambda,\mu,v} z^k = \frac{\Gamma(k+1)\Gamma(1-\mu+v+k)}{\Gamma(1-\mu+k)\Gamma(1-\lambda+v+k)} z^{k-\mu} \qquad \ldots 3.5$$

$(0 \le \lambda < 1, \mu, v \in IR \;\; \& \; k > \max\{0, \mu - v\} - 1)$

2. Coefficient Bounds:

Lemma 2.1: If $f(z) \in T(n,p)$ **then** $f(z) \in \sum_p^{\lambda,\mu}(\gamma,\beta)$ **iff**

$$\sum_{k=n+p}^{\infty} [\gamma k - \gamma + 1)(k-\beta)]) \, C_p^{\lambda,\mu}(k) a_k < (\gamma p - \gamma + 1)(p-\beta) \qquad \ldots 2.1$$

Where $0 \le \beta < p, \; 0 \le \gamma \le 1, p \in IN \;\; \& C_p^{\lambda,\mu}(k)$ **is given by,**

$$C_p^{\lambda,\mu}(k) = \frac{\Gamma(k-p+1+\mu)\Gamma(v+2+\mu-\lambda)\Gamma(k+\cup-p+2)}{\Gamma(k-p+1)\Gamma(k+v-p+2+\mu-\lambda)\Gamma(v+2)\Gamma(1+\mu)} \qquad \ldots 2.1A$$

Proof: we have,

$f \in \sum_p^{\lambda,\mu}(\gamma,\beta)$ **By using,**

$$\vartheta_p^{\lambda,\mu} f(z) = \frac{\Gamma(\mu-\lambda+v+2)}{\Gamma(v+2)\Gamma(\mu+1)} Z^p J_{0,z}^{\lambda,\mu,v}(z^{\mu-p} f(z))$$

$$= z^p - \sum_{k=n+p}^{\infty} a_k C_p^{\lambda,\mu}(k) z^k$$

$$\left(\vartheta_p^{\lambda,\mu} f(z)\right)' = p z^{p-1} - \sum_{k=n+p}^{\infty} k a_k C_p^{\lambda,\mu}(k) z^{k-1} \qquad \ldots 2.11$$

And

$$\left(\vartheta_p^{\lambda,\mu} f(z)\right)'' = p(p-1) z^{p-2} - \sum_{k=n+p}^{\infty} k(k-p) a_k C_p^{\lambda,\mu}(k) z^{k-2} \qquad \ldots 2.12$$

But with the help of above two equations, in

$$R_e\left\{\frac{z\left(\vartheta_p^{\lambda,\mu} f(z)\right)' + \Gamma Z^2\left(\vartheta_p^{\lambda,\mu} f(z)\right)''}{\Gamma^2\left(\vartheta_p^{\lambda,\mu} f(z)\right)' + (1-\gamma)((\vartheta_p^{\lambda,\mu} f(z))}\right\} > \beta \qquad \ldots 2.13$$

We show that,

$$R_e\left\{\frac{pz^p - \sum_{k=n+p}^{\infty} ka_k C_p^{\lambda,\mu}(k)z^k + \gamma p(p-1)z^p - \sum_{k=n+p}^{\infty} \gamma k(k-1)a_k C_p^{\lambda,\mu}(k)z^k}{\gamma pz^p - \sum_{k=n+p}^{\infty} \gamma ka_k C_p^{\lambda,\mu}(k)z^k + (1-\gamma)(z^p - \sum_{k=n+p}^{\infty} a_k C_p^{\lambda,\mu}(k)z^k)}\right\} > \beta$$

Then $z - 1^-$ through real values,

We have,

$$(1-\gamma+\gamma p)\beta - \sum_{k=n+p}^{\infty} (\gamma k - \gamma + 1)\beta a_k c_p^{\lambda,\mu}(k) \le (\gamma p(p-1) + p) - \sum_{k=n+p}^{\infty} (k + \gamma k(k-1))a_k c_p^{\lambda,\mu}(k) \ldots\ldots\ldots 2.14$$

Therefore,

$$\sum_{k=n+p}^{\infty} [(\gamma k - \gamma + 1)(k-\beta)]c_p^{\lambda,\mu}(k)a_k \le (\gamma p - \gamma + 1)(p - \beta)$$

Conversely, by using above lemma show that (2.13) is satisfied and so $f \in \sum_p^{\lambda,\mu}(\gamma,\beta)$ **since** $Re\{w\} > \beta$ **iff** $|w - (1+\beta)| < |w + (1-\beta)|$ **it is sufficient to prove that,**

$$Q = \left|\frac{z(\vartheta_p^{\lambda,\mu} f(z)' + \gamma z^2(\vartheta_p^{\lambda,\mu} f(z)"}{\gamma z(\vartheta_p^{\lambda,\mu} f(z)' + (1-\gamma)(\vartheta_p^{\lambda,\mu} f(z))} - 1 - \beta\right| < \left|\frac{z(\vartheta_p^{\lambda,\mu} f(z)' + \gamma z^2(\vartheta_p^{\lambda,\mu} f(z)"}{\gamma z(\vartheta_p^{\lambda,\mu} f(z)' + (1-\gamma)(\vartheta_p^{\lambda,\mu} f(z))} + 1 - \beta\right| \ldots\ldots\ldots\ldots\ldots 2.15$$

=T

Let, $X = \gamma z(\vartheta_p^{\lambda,\mu} f(z)` + (1-\gamma)(\vartheta_p^{\lambda,\mu} f(z)$ **then we have,**

$$Q = \frac{1}{1X1}\left|z(\vartheta_p^{\lambda,\mu} f(z)' + \gamma z^2(\vartheta_p^{\lambda,\mu} f(z)" - (1+\beta)X\right|$$

From (I), 2.11, 2.12 we get,

$$Q = \frac{1}{|X|}|[(p + \gamma p(p-1) - (1+\beta)\gamma p - (1+\beta)(1-\gamma)|z^p - \sum_{K=n+p}^{\infty}[(k + \Gamma k(k-1)) - (1+\beta)(\gamma k - \gamma + 1)]c_p^{\lambda,\mu}(k)a_k z^k|$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2.16$$

$$= \frac{1}{|X|}|[p(1-(1+\beta)\gamma) - (1+\beta)(1-\gamma) + \gamma p(p-1)]z^p - \sum_{K=n+p}^{\infty}[(\gamma k - \gamma + 1)(k - \beta - 1)]c_p^{\lambda,\mu}(k)a_k z^k|$$

$$< \frac{|z|^p}{|x|}[(\gamma p - \gamma + 1)(p - \beta - 1) + \sum_{k=n+p}^{\infty}(\gamma k - \gamma + 1)(k - \beta - 1)\, c_p^{\lambda,\mu}(k)a_k|z|^{k-p}]$$

And,

$$T = \frac{1}{|X|}\left|z(\vartheta_p^{\lambda,\mu} f(z)' + \gamma z^2(\vartheta_p^{\lambda,\mu} f(z)" - (1-\beta)X\right|$$

$$= \frac{1}{|X|}|[p(1-(1-\beta)\gamma)-(1-\beta)(1-\gamma)+\gamma p(p-1)]z^p - \sum_{K=n+p}^{\infty}[(\gamma k-\gamma+1)(k-\beta+1)]c_p^{\lambda,\mu}(k)a_k z^k|$$

$$\geq \frac{|z|^p}{|x|}[(\gamma p-\gamma+1)(p-\beta+1)+\sum_{k=n+p}^{\infty}(\gamma k-\gamma+1)(k-\beta+1)\,c_p^{\lambda,\mu}(k)a_k\,|z|^{k-p}] \quad \ldots\ldots\ldots\ldots 2.17$$

As, $z \in \partial \cup = \{z: z \in \mathbb{C}\ and\ |z| = 1\}$
Simply we shoe that $T - Q > 0$ **as (2.1) condition satisfied so the above lemma proved.**

Remark: 1 The lemma (2.1) is sharp to f(z) function which,

$$f(Z) = z^p - \frac{(\gamma p-\gamma+1)(p-\beta)}{\gamma(n+p-1)(n+p-\beta)c_p^{\lambda,\mu}(n+p)}Z^{n+p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2.18$$

$$c_p^{\lambda,\mu}(n+p) = \frac{\Gamma(n+1+\mu)\Gamma(v+2+\mu-\lambda)\Gamma(n+v+2)}{\Gamma(n+1)\Gamma(n+v+2+\mu-\lambda)\Gamma(v+2)\Gamma(1+\mu)}$$

Corollary: 1 If $f(z) \in \sum_p^{\lambda,\mu}(\gamma,\beta)$ **then** $a_k \leq \frac{(\gamma p-\gamma+1)(p-\beta)}{(\gamma k-\gamma+1)(k-\beta)c_p^{\lambda,\mu}(k)}; (k \geq n+p; n \in IN)$ ……………………… 2.19

Where $c_p^{\lambda,\mu}(k)$ **is shown by equation (2.1A)**

3. Radii of starlikeness and convexity;

Lemma: 3.1 consider $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$, $g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k$

Be in $\sum_p^{\lambda,\mu}(\gamma,\beta)$ **then the function,**

$h(z) = z^p - \sum_{k=n+p}^{\infty}(a_k^m + b_k^m)z^k$; $m \in IN$ **Is also in** $\sum_p^{\lambda,\mu}(\gamma,\beta)$**, where;**

$$\beta_1 < inf_k\left\{\frac{[p(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(v+2+\mu-\lambda)\Gamma(k+v-p+2)]^{m-1}(k-\beta)^m - 2k[(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+v-p+2+\mu-\lambda)\Gamma(v+2)\Gamma(1+\mu)]^{m-1}(p-\beta)^m]}{[[(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(v+2+\mu-\lambda)\Gamma(k+v-p+2)]^{m-1}(k-\beta)^m - 2[(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+v-p+2+\mu-\lambda)\Gamma(v+2)\Gamma(1+\mu)]^{m-1}(p-\beta)^m]}\right\}$$

…………3.11

Proof: since f, g **belong to** $\sum_p^{\lambda,\mu}(\gamma,\beta)$ **then,**
We have,

$$\sum_{k=n+p}^{\infty}\left[\frac{(k-\beta)(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(v+2+\mu-\lambda)\Gamma(k+v-p+2)}{(p-\beta)(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+v-p+2+\mu-\lambda)\Gamma(v+2)\Gamma(1+\mu)}\right]^m a_k^m$$

$$\leq \sum_{k=n+p}^{\infty}\left[\frac{(k-\beta)(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(v+2+\mu-\lambda)\Gamma(k+v-p+2)}{(p-\beta)(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+v-p+2+\mu-\lambda)\Gamma(v+2)\Gamma(1+\mu)}a_k\right]^m < 1$$

$$\sum_{k=n+p}^{\infty}\left[\frac{(k-\beta)(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(v+2+\mu-\lambda)\Gamma(k+v-p+2)}{(p-\beta)(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+v-p+2+\mu-\lambda)\Gamma(v+2)\Gamma(1+\mu)}\right]^m b_k^m$$
$$\leq\left[\sum_{k=n+p}^{\infty}\frac{(k-\beta)(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(v+2+\mu-\lambda)\Gamma(k+v-p+2)}{(p-\beta)(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+v-p+2+\mu-\lambda)\Gamma(v+2)\Gamma(1+\mu)}b_k\right]^m<1$$

As a result,

$$\frac{1}{2}\sum_{k=n+p}^{\infty}\left[\frac{(k-\beta)(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(v+2+\mu-\lambda)\Gamma(k+v-p+2)}{(p-\beta)(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+v-p+2+\mu-\lambda)\Gamma(v+2)\Gamma(1+\mu)}\right]^m (a_k^m+b_k^m)<1$$

Then we shows that,

$$\sum_{k=n+p}^{\infty}\left[\frac{(k-\beta_1)(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(v+2+\mu-\lambda)\Gamma(k+v-p+2)}{(p-\beta_1)(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+v-p+2+\mu-\lambda)\Gamma(v+2)\Gamma(1+\mu)}\right]^m (a_k^m+b_k^m)<1$$

But inequality holds true if,

$$\left[\frac{(k-\beta_1)(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(v+2+\mu-\lambda)\Gamma(k+v-p+2)}{(p-\beta_1)(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+v-p+2+\mu-\lambda)\Gamma(v+2)\Gamma(1+\mu)}\right]$$
$$\leq\frac{1}{2}\left[\frac{(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(v+2+\mu-\lambda)\Gamma(k+v-p+2)}{(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+v-p+2+\mu-\lambda)\Gamma(v+2)\Gamma(1+\mu)}\right]^m\left(\frac{k-\beta}{p-\beta}\right)^m$$

Or $\beta_1\leq\frac{w_p-k}{w-1}$

As the above lemma is proved by W which is right hand side of above inequality.

Lemma: 3.2 If $f(z)\in\sum_p^{\lambda,\mu}\gamma,\beta$ **then integrals operator** $F_i(z)=(1-i)z^p+ip\int_0^z\frac{f(s)}{s}ds\ (i\geq 0, z\in\cup)$

Also in $\sum_p^{\lambda,\mu}\gamma,\beta\ 0\leq i\leq\frac{n+p}{p}$

Proof: let, $f(z)=z^p-\sum_{z=n}^{\infty}a_k z^k$ **then**

$$F_i(z)=(1-i)z^p+ip\int_0^z\left(\frac{s^p\sum_{k=n+p}^{\infty}a_k s^k}{s}\right)ds$$
$$=(1-i)z^p+ip\left[\frac{1}{p}z^p-\sum_{k=n+p}^{\infty}\frac{a_k}{k}z^k\right]$$
$$=z^p-\sum_{k=n+p}^{\infty}\frac{ip}{k}a_k z^k=z^p-\sum_{k=n+p}^{\infty}g_k z^k,$$

Where, $g_k=\frac{ip}{k}a_k$ **but,**

$$\sum_{k=n+p}^{\infty}\frac{(k-\beta)(\gamma\beta-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(v+2+\mu-\lambda)\Gamma(k+v-p+2)}{\Gamma(k-p+1)\Gamma(k+\gamma-p+2+\mu-\lambda)\Gamma(v+2)\Gamma(1+\mu)}g_k$$

$$= \sum_{k=n+p}^{\infty} \frac{(k-\beta)(\gamma\beta-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(v+2+\mu-\lambda)\Gamma(k+v-p+2)}{\Gamma(k-p+1)\Gamma(k+v-p+2+\mu-\lambda)\Gamma(v+2)\Gamma(1+\mu)} \frac{ip}{k} a_k$$

$$\leq \sum_{k=n+p}^{\infty} \frac{(k-\beta)(\gamma\beta-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(v+2+\mu-\lambda)\Gamma(k+v-p+2)}{\Gamma(k-p+1)\Gamma(k+v-p+2+\mu-\lambda)\Gamma(v+2)\Gamma(1+\mu)} \frac{ip}{n+p} a_k$$

$$\left(\frac{ip}{n+p} \leq 1\right) \leq \sum_{k=n+p}^{\infty} \frac{(k-\beta)(\gamma\beta-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(v+2+\mu-\lambda)\Gamma(k+v-p+2)}{\Gamma(k-p+1)\Gamma(k+v-p+2+\mu-\lambda)\Gamma(v+2)\Gamma(1+\mu)} a_k \leq (p-\beta)(\gamma p-\gamma+1)$$

Remark: 2$\sigma(0 \leq \sigma < 1) for\ G(Z) and\ F(Z)$ **is radii of starlikeness and convexity,**

$$G(z): \begin{cases} (i) r_1 = inf_k \left\{ \frac{(c+k)(p-\sigma)(k-\beta)(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(v+2+\mu-\lambda)\Gamma(k+v-p+2)}{(c+p)(k-\sigma)(p-\beta)(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+v-p+2+\mu-\lambda)\Gamma(v+2)\Gamma(1+\mu)} \right\}^{1/k-p} \\ (ii)\ r_2 = inf_k \left\{ \frac{p(c+k)(p-\sigma)(k-\beta)(\gamma k-v+1)\Gamma(k-p+1+\mu)\Gamma(v+2+\mu-\lambda)\Gamma(k+v-p+2)}{k(c+p)(k-\sigma)(p-\beta)(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+v-p+2+\mu-\lambda)\Gamma(v+2)\Gamma(1+\mu)} \right\}^{1/k-p} \end{cases} \dots\dots 3.12$$

$$F_i(z): \begin{cases} (i) R_1 = inf_k \left\{ \frac{k(p-\sigma)(k-\beta)(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(v+2+\mu-\lambda)\Gamma(k+v-p+2)}{ip(k-\sigma)(p-\beta)(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+v-p+2+\mu-\lambda)\Gamma(v+2)\Gamma(1+\mu)} \right\}^{1/k-p} \\ (ii) R_1 = inf_k \left\{ \frac{(p-\sigma)(k-\beta)(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(v+2+\mu-\lambda)\Gamma(k+v-p+2)}{i(k-\sigma)(p-\beta)(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+v-p+2+\mu-\lambda)\Gamma(v+2)\Gamma(1+\mu)} \right\}^{1/k-p} \end{cases} \dots\dots\dots\dots 3.13$$

4. Quasi-Hadamard Product:-

Definition 4.1: Let, f_j**(z) (j=1,……..,m) in the class of** $\sum_p^{\lambda,\mu}(\gamma,\beta)$ **define,**

$$f_j(z) = z^p - \sum_{k=n+1}^{\infty} a_{k,j} z^k \quad (j = 1, \dots\dots\dots, m; n, p \in IN = \{1,2, \dots\dots\})$$

Then the Quasi-Hadamard product of the function $f_j(z)$ **denoted by** $(f_1 * f_2 * f_3 * \dots\dots\dots * f_3)(z)$ **defined by,**

$$(f_1 * f_2 * f_3 * \dots\dots\dots * f_3)(z) = z^p - \sum_{k=n+p}^{\infty} (a_{k,1} a_{k,2} \dots\dots\dots\dots . a_{k,m})\, z^k \dots\dots\dots\dots\dots\dots\dots 4.1$$

Lemma 4.1: Let $f_j(z) \in \sum_p^{\lambda,\mu}(\gamma,\beta_j)$ **then** $(f_1 * f_2 * f_3 * \dots\dots\dots * f_n)(z) \in \sum_p^{\lambda,\mu}(\gamma,w)$ **where** $0 < w \leq p - \frac{n}{H(n+p,m)-1}$**…4.11**

And $H(n+p,m) = \left[\frac{(\gamma(n+p-1)-1)c_p^{\lambda,\mu}(n+p)(n+p-w)}{(\gamma p-\gamma+1)(p-w)}\right]^{m-1} \prod_{j=1}^{m}\left(\frac{n+p-\beta_j}{p-\beta_j}\right.$ **……………………..4.12**

The proof of the above lemma is sharp for the function $f_j(z)(j = 1,2,3, \dots\dots\dots\dots m) as,$

$$f_j(z) = z^p - \frac{\gamma p - \gamma + 1)(p - \beta_j)}{(\gamma(n+p-1)(n+p-\beta_j)c_p^{\lambda,\mu}(n+p)} z^{n+p} \dots\dots\dots\dots\dots 4.13$$

Proof: By induction method m=1,w= $\beta,\ m = 2$ **the inequality (2.1) shows,**

$$\sum_{k=n+p}^{\infty} \frac{[(\gamma k - \gamma + 1)(k - \beta_j)]\ c_p^{\lambda,\mu}(k)}{(\gamma p - \gamma + 1)(p - \beta_j)} a_{k,j} \le 1 \quad (j = 1,2; n, p \in IN)$$

Thus,

$$\sum_{k=n+p}^{\infty} \frac{(\gamma k - \gamma + 1)}{(\gamma p - \gamma + 1))} c_p^{\lambda,\mu}(k) \sqrt{\prod_{j=1}^{2} \frac{(k - \beta_j)}{(p - \beta_j)}} a_{k,j} \le 1 \qquad \dots\dots\dots\dots\dots 4.14$$

Now the largest w **value is given,**

$$\sum_{k=n+p}^{\infty} \frac{[(\gamma k - \gamma + 1)}{(\gamma p - \gamma + 1))} c_p^{\lambda,\mu}(k)\, a_{k,1} a_{k,2} \le 1$$

Or $\frac{k-w}{p-w}\sqrt{a_{k,1}a_{k,2}} \le \sqrt{\prod_{j=1}^{2}\frac{(k-\beta_j)}{(p-\beta_j)}}$ $\qquad k \ge n + p$ **using (4.14) 4.15**

$$\frac{k - w}{p - w} \le \frac{(\gamma k - \gamma + 1)}{(p\gamma - \gamma + 1)} c_p^{\lambda,\mu}(k) \prod_{j=1}^{2} \frac{(k - \beta_j)}{(p - \beta_j)}$$

$$\frac{k - w}{p - w} \le y(k) \quad as,$$

$$Y(k) = \frac{[(\gamma k - \gamma + 1)}{(p\gamma - \gamma + 1)} c_p^{\lambda,\mu}(k) \prod_{j=1}^{2} \left(\frac{k - \beta_j}{p - \beta_j}\right)$$

Or $k - w \le (p - w)Y(f)\ \ or\ 0 < w \le p - \frac{n}{Y(k)-1}$ $\qquad \dots\dots\dots\dots 4.16$

By function $\psi(k)$ **by** $\psi(k) = p - \frac{n}{Y(k)-1}$ $\qquad k \ge n + p$

Since $\psi'(k) = \frac{nY'(k)}{(Y(k)-1)^2} \ge 0$ **for** $k \ge n + p$

So $\psi(k)$ **is an increasing function and** $0 < w \le \psi(n + p) \le p - \frac{n}{Y(n+p)-1}$

Where,

$Y(n + p) = \frac{((n+p)\gamma - \gamma + 1}{(p\gamma - \gamma + 1)} c_p^{\lambda,\mu}(n + p) \prod_{j=1}^{2}\left(\frac{(n+p)-\beta_j}{p-\beta_j}\right)$ $\qquad \dots\dots\dots\dots\dots 4.17$

Therefore the result is true for m=2 let we assume the above result is true for fixed natural number m, m+1, that is $(f_1 * f_2 * f_3 * \dots\dots\dots * f_m * f_{m+1})(z) \in \sum_p^{\lambda,\mu}(\gamma,\ \eta)$

Where η **satisfied condition** $o < \eta < p - \frac{n}{H-1}$ $\qquad \dots\dots\dots\dots\dots 4.18$

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And $H = \frac{(n+p)(\gamma-\gamma+1)c_p^{\lambda,\mu}(n+p)(n+p-w)(n+p-\beta_m+1)}{(p\gamma-\gamma+1)(p-w)(p-\beta_m+1)}$

This is shown by (4.11) also $0 < \eta < p - \frac{n}{H(n+p,m+1)-1}$ **.**

The above form shows m+1. Therefore by induction method we obtain that m as positive integer hence, the result is true.

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